MAC 2312 – Calculus II Guided Notes

Section 11.10 – part 2

The Binomial Series

Definition

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

$|x| < 1$

Recall $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}$

or $\binom{k}{n} = \frac{k!}{(k-n)!\,n!}$

Example 1

Find a power series representation for $\sqrt[3]{1+x}$

$(1+x)^{1/3} \qquad k = \frac{1}{3}$

$$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \cdots$$

$$+ \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\cdots\left(\frac{1}{3}-n+1\right)}{n!}x^n + \cdots$$

so cleaned up

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x - \frac{2}{3^2(2!)}x^2 + \frac{1\cdot 2\cdot 5}{3^3(3!)}x^3 + \cdots + (-1)^{n+1}\frac{1\cdot 2\cdots(3n-4)}{3^n(n!)}x^n + \cdots$$

$|x|<1$
$n \geq 2$

Example 2 Find a power series representation for $\sqrt[3]{1+x^4}$

substitute x^4 in place of x in example 1

so

$$\sqrt[3]{1+x^4} = 1 + \frac{1}{3}x^4 - \frac{2}{3^2\cdot 2!}x^8 + \cdots + (-1)^{n+1}\frac{1\cdot 2\cdots(3n+4)}{3^n\cdot n!}x^{4n} + \cdots$$

Example 3 approximate $\int_0^3 \sqrt[3]{1+x^4}\,dx$ accurate to six decimal places.

Use the result from example 2

$$\int_0^3 \sqrt[3]{1+x^4}\,dx = 0.3 + 0.000162 - \underbrace{0.000000243}_{6 \text{ zeros}} + \cdots \simeq \boxed{0.300162}$$

note that $\sqrt[3]{1+x^4} \simeq 1 + \frac{1}{3}x$

Example 4

Find a power series representation for f(x) and state the radius of convergence.

$$f(x) = x(1+2x)^{-2}$$

start with

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots$$

replacements

$x \to 2x$

$k = -2$

then multiply the result by x

$$x(1+2x)^{-2} = x\left(1 + -2(2x) + \frac{(-2)(-3)}{2!}(2x)^2 + \cdots + \frac{(-2)(-3)\cdots(-2-n-1)}{n!}(2x)^n + \cdots\right)$$

$$= x - 4x^2 + 3(4x^2)(x) + \cdots + \frac{(-2)(-3)\cdots(-2-n-1)}{n!}(4x^n)x + \cdots$$

$$= \boxed{x - 4x^2 + 12x^3 + \cdots + \frac{(-2)(-3)\cdots(-2-n-1)}{n!}(4x^{n+1}) + \cdots}$$

radius of convergence

$$|2x| < 1$$

$$|x| < \frac{1}{2}$$

Section 11.10 — part 2 Practice Problems

Binomial Expansion

Find the power series representation for $f(x)$ and radius of convergence.

1. $f(x) = \dfrac{1}{\sqrt[3]{1+x}}$

2. $f(x) = \dfrac{1}{\sqrt[3]{1-x^2}}$

3. $f(x) = (4+x)^{3/2}$

4. Evaluate $\displaystyle\int_0^{1/2} \frac{1}{\sqrt[3]{1+x^2}}\,dx$ accurate to three decimal places.

Section 11.10 part 2 Binomial Expansion	answers & solution hints

① $k = -\frac{1}{3}$ $\boxed{r=1}$

$$\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{1}{3}x + \frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}x^2 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n n!} x^n$$

② $k = -\frac{1}{3}$ replace x with $-x^2$ $r=1$

$$\frac{1}{\sqrt[3]{1-x^2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{3^n \cdot n!} x^{2n}$$

③ $(4+x)^{3/2} = (\sqrt{4+x})^3 = \left[2\sqrt{1+(\frac{x}{2})}\right]^3$

$= 8(1+\frac{x}{2})^{3/2}$ so $k = \frac{3}{2}$ $x \to \frac{x}{2}$

$$(4+x)^{3/2} = 8\left[1 + \frac{3}{2}\left(\frac{x}{4}\right) + \frac{(\frac{3}{2})(\frac{1}{2})}{2!}\left(\frac{x}{4}\right)^2 + \cdots\right]$$

$$= 8 + 3x + \frac{3}{16}x^2 + \sum_{n=3}^{\infty} (-1)^{n-1} \frac{(1 \cdot 3 \cdot 5 \cdots (2n-5)}{8^n \cdot n!} x^n$$

Converges when $\left|\frac{x}{4}\right| < 1$ or $|x| < 4$

so $r = 4$

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4 $$\int_0^{1/2} \frac{1}{\sqrt[3]{1+x^2}}\,dx = \int_0^{1/2} (1+x^2)^{-1/3}\,dx$$ $k = -\frac{1}{3}$
replace x
with x^2

$$\int_0^{1/2} \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n \cdot n!} x^{2n} \right] dx$$

$$\approx \frac{1}{2} - \frac{1}{72} + \frac{1}{720} - \frac{1}{5184} \approx \boxed{0.487}$$

$\boxed{\text{from}}$ $$\frac{1}{2} - \frac{1}{3\cdot 2^3\cdot 3} + \frac{1\cdot 4}{9\cdot 2\cdot 2^5\cdot 5} - \frac{1\cdot 4\cdot 7}{3^3\cdot 6\cdot 2^7\cdot 7}$$